Spectral density

1. $\mathcal{O}(x) = \phi(x)$

For the case at hand only 1-particle states contribute

$|\vec{p}, r = M^2\rangle$, r is mass^2 of the state

$$\langle \vec{p}\,', M'^2 | \vec{p}, M^2\rangle = (2\pi)^3\, 2E_{p,M}\, \delta(M^2 - M'^2)\, \delta(\vec{p} - \vec{p}\,')$$

Matrix element:

$$\langle 0 | \phi(0) | \vec{p}, m^2\rangle = \sqrt{Z_\phi} = 1$$

There is only one value for mass $M^2 = m^2$, therefore,

$$\rho_\phi(\mu^2) = \int dr\, \delta(\mu^2 - m_r^2) = \delta(\mu^2 - m^2)$$

$$\phi(0) = \int d\Omega_k\, (a_k + a_k^\dagger)$$

$$\langle 0 | \phi(0) | \vec{p}, m\rangle$$

$$\langle 0 | \int d\Omega_k\, (\underbrace{a_k + a_k^\dagger)\, a_p^\dagger}_{(2\pi)^3\, 2E_k\, \delta(\vec{p} - \vec{k})} | 0\rangle = 1$$

2. In this case only 2-particle states contribute:

$\rho_0(\mu^2) = \int |Z_{0,r}|\, \delta(\mu^2 - m_r^2)\, dr$

$\langle 0 | O(0) | \vec{p}, \underbrace{m_r^2}_{r} \rangle$

$$\langle 0 | \phi^2(0) | \vec{p}_1, \vec{p}_2 \rangle$$

$$= \int d\Omega_1\, d\Omega_2 \langle 0 | (a_{k_1} + a_{k_1}^+)(a_{k_2} + a_{k_2}^+) | \vec{p}_1, \vec{p}_2 \rangle = 2$$

However, we need to consider states with fixed angular momentum (only states with 0 angular momentum)

$P^2 = r = m_r^2$

$$|\vec{p}, r\rangle = C(\vec{p}, r) \int \delta(r - (E_1+E_2)^2 + (\vec{p}_1 + \vec{p}_2)^2)\, \delta(\vec{p}_1 + \vec{p}_2 - \vec{P})$$
$$\times\, d\Omega_1\, d\Omega_2\, |\vec{p}_1, \vec{p}_2\rangle$$

Let's compute $Z_{\phi^2, r}$:

$$\sqrt{Z_{\phi^2, r}} = \langle 0 | \phi^2(0) | \vec{p}, r \rangle = \langle 0 | \phi^2(0) | 0, r \rangle$$

$$= 2C(0, r) \int \delta(r - P^2)\, \delta(\vec{P})\, d\Omega_1\, d\Omega_2$$

$\vec{P} \equiv \vec{p}_1 + \vec{p}_2$, $P = (E_1 + E_2, \vec{P})$

Coefficient $c(\vec{p}, r)$ is fixed from

$$\langle \vec{p}', r' | \vec{p}, r \rangle = (2\pi)^3\, 2E_{p,r}\, \delta(r - r')\, \delta(\vec{p} - \vec{p}')$$

$$\langle \vec{p}', r' | \vec{p}, r \rangle = C(\vec{p}', r')\, C(\vec{p}, r)$$

$$\times \int d\Omega_1 d\Omega_2 d\Omega_1' d\Omega_2' \delta(r - P^2)\, \delta(r' - P'^2)\, \delta(\vec{P} - \vec{p})\, \delta(\vec{P}' - \vec{p}')$$

$$\times \langle \vec{p}_1', \vec{p}_2' | \vec{p}_1, \vec{p}_2 \rangle$$

$$= 2C(\vec{p}', r')\, c(\vec{p}, r) \int d\Omega_1 d\Omega_2\, \delta(\vec{P} - \vec{p}')\, \delta(\vec{P} - \vec{p})$$

$$\times\, \delta(r - P^2)\, \delta(r' - P^2)$$

$$= 2\, c^2(\vec{p}, r)\, \delta(r - r')\, \delta(\vec{p} - \vec{p}') \int \delta(\vec{P} - \vec{p})\, \delta(r - P^2)\, d\Omega_1 d\Omega_2$$

Comparing, we get:

$$c^2(0, r) \underbrace{\int d\Omega_1 d\Omega_2\, \delta(r - P^2)\, \delta(\vec{P})}_{I(r)} = (2\pi)^3 \sqrt{r}$$

m_r

$r = m_r^2$

$$c^2(0, r) = \frac{(2\pi)^3 \sqrt{r}}{I(r)}$$

Hence,

$$\rho_{\phi^2}(\mu^2) = \int \left[2\, c(0, r)\, I(r)\right]^2 \delta(\mu^2 - r)\, dr$$

$$= 4\, c^2(0, \mu^2)\, I^2(\mu^2)$$

$$= 4\, I^2(\mu^2)\, \frac{(2\pi)^3 \mu}{I(\mu^2)} = 4\mu\, (2\pi)^3\, I(\mu^2)$$

Lastly, we compute the integral:

$$I(\mu^2) = \int d\Omega_1\, d\Omega_2\, \delta(\mu^2 - P^2)\, \delta(\vec{P})$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}\, \delta\left(\mu^2 - (E_1 + E_2)^2 + (\vec{p}_1 + \vec{p}_2)^2\right) \delta(\vec{p}_1 + \vec{p}_2)$$

$$= \int \frac{d^3 p}{(2\pi)^6 (2E_{p,m})^2}\, \delta(\mu^2 - 4E_{p,m}^2) = \frac{\pi}{(2\pi)^6\, 4\mu} \sqrt{1 - \frac{4m^2}{\mu^2}}$$

Leading to

$$\rho_{\phi^2}(\mu^2) = \frac{1}{8\pi^2} \sqrt{1 - \frac{4m^2}{\mu^2}}$$

$$\tilde{D}_{\phi^2}(p^2) = \int d\mu^2\, \frac{i}{p^2 - \mu^2 + i\varepsilon}\, \rho(\mu^2)$$

3. • $\langle 0| J^\mu(x)\, \phi(y) |0\rangle$

$$= \sum_{\vec{p}, S_3, r} e^{-ip_r(x-y)} \langle 0| J^\mu(0) |\vec{p}, S_3; r\rangle \langle \vec{p}, S_3; r| \phi(0) |0\rangle$$

$$= -i \sum_{\vec{p}, r} e^{-ip_r(x-y)}\, p_r^\mu f_r c_r^*$$

• $\langle 0| \phi(y)\, J^\mu(x) |0\rangle$

$$= \sum_{\vec{p}, S_3, r} e^{ip_r(x-y)} \langle 0| \phi(0) |\vec{p}, S_3; r\rangle \langle \vec{p}, S_3; r| J^\mu(0) |0\rangle$$

$$= -\sum_{n} e^{ip_r(x-y)} \langle 0| \Theta^+ \phi(0) \Theta |n\rangle \langle n| \Theta^+ J^\mu(0) \Theta |0\rangle$$

$$= -\sum_{n} e^{ip_r(x-y)} \langle \Theta 0| \phi(0) |\Theta n\rangle^* \langle \Theta n| J^\mu(0) |\Theta 0\rangle^* =$$

$$= -\sum_{n} e^{ip_r(x-y)} \langle \Theta n| \phi(0) |0\rangle \langle 0| J^\mu(0) |\Theta n\rangle$$

$$= -\sum_{\vec{p}, S_3, r} e^{ip_r(x-y)} \langle 0| J^\mu(0) |\vec{p}, S_3; r\rangle \langle \vec{p}, S_3; r| \phi(0) |0\rangle$$

$$= i \sum_{\vec{p}, r} e^{ip_r(x-y)}\, p_r^\mu f_r c_r^*$$

- $\langle 0 | T\, J^\mu(x)\, \phi(y) \,| 0 \rangle$

$$= -i \sum_{\vec{p},\, r} p_r^\mu f_r c_r^* \times \left[\theta(x_0 - y_0)\, e^{-ip_r(x-y)} - \theta(y_0 - x_0)\, e^{ip_r(x-y)} \right]$$

$$= \int dr \int \frac{d^3 p}{(2\pi)^3\, 2E_p}\, \partial_x^\mu f_r c_r^* \times \left[\theta(x_0 - y_0)\, e^{-ip_r(x-y)} + \theta(x_0 - y_0)\, e^{ip_r(x-y)} + \text{contact terms} \right]$$

$$= \int dr\, f_r c_r^*\, \partial_x^\mu G^{(m_r)}(x-y)$$

- Last eq,

$$\partial_\mu^x \langle 0 | T\, J^\mu(x)\, \phi(y) \,| 0 \rangle = -if\, \delta^{(4)}(x-y)$$

$$\partial_x^2 G + m^2 G = \delta(x-y)$$

$$\partial_\mu^x \int dr\, f_r c_r^* \partial_x^\mu \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m_r^2 + i\varepsilon}$$

$$= -\int dr\, f_r c_r^* \int \frac{d^4p}{(2\pi)^4} \frac{i p^2 e^{-ip(x-y)}}{p^2 - m_r^2 + i\varepsilon}$$

Fourier transform:

$$\int dr\, f_r c_r^* \frac{p^2}{p^2 - m_r^2 + i\varepsilon} = f,$$

which can be satisfied if and only if $m_r = 0$, otherwise the l.h.s. is p-dependent